Solution to Problem Set 13
Optical Waveguides and Fibers (OWF)

Exercise 1: Diagonalization of the coupled-mode equation

When two single-mode waveguides are brought in close proximity to each other (Fig. 1) the mode profile of one waveguide can extend into the region occupied by the other waveguide. This leads to an interaction of the two original modes which can be described by the following coupled-mode equations:

$$\begin{aligned} \frac{\partial A_1}{\partial z} &= -j\kappa_{12} A_2 e^{-j(\beta_2-\beta_1)z}, \\ \frac{\partial A_2}{\partial z} &= -j\kappa_{21} A_1 e^{-j(\beta_1-\beta_2)z}. \end{aligned} \tag{1}$$

In these relations, A_ν is the mode amplitude in waveguide ν, $\nu \in \{1,2\}$ and the coupling coefficients $\kappa_{\nu\mu}$, $\mu \in \{1,2\}$ and $\mu \neq \nu$ are defined by:

$$\kappa_{\nu\mu} = \frac{\omega}{4P_\nu} \int\int \Delta\epsilon_\nu(x,y) \underline{\mathcal{E}}_\mu(x,y) \underline{\mathcal{E}}_\nu^*(x,y) \mathrm{d}x\mathrm{d}y, \tag{2}$$

where $\Delta\epsilon_\nu(x,y)$ is the perturbation of the background dielectric constant which defines the waveguide ν, and where $\underline{\mathcal{E}}_\mu(x,y)$ ($\underline{\mathcal{E}}_\nu(x,y)$) is the mode field distribution of the waveguide μ (ν) assuming that the respective other waveguide is absent. Assume that the two waveguides are identical, i.e., $\beta_1 = \beta_2$ and $\kappa_{12} = \kappa_{21} = \kappa$.

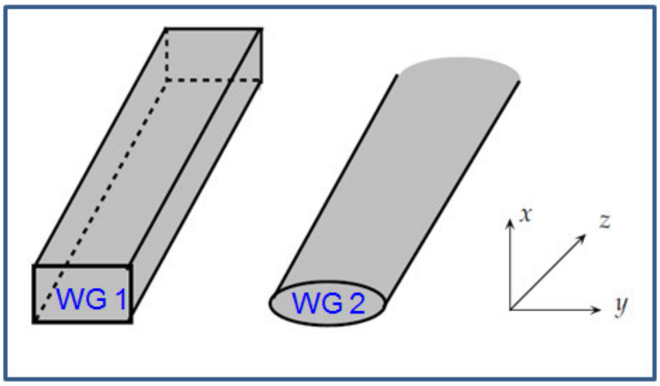

Figure 1: Two waveguides coupled to each other.

a) Write Eq. (1) and Eq. (2) in the following matrix form:

$$\frac{\partial}{\partial z}\mathbf{A}(z) = -j\mathbf{M}_A\mathbf{A}(z) \tag{3}$$

where $\mathbf{A}(z) = (A_1(z),\, A_2(z))^T$.

Solution: Plugging-in $\beta_1 = \beta_2$ and $\kappa_{12} = \kappa_{21} = \kappa$ into Eq. (1), we get:

$$\begin{aligned} \frac{\partial}{\partial z} A_1(z) &= -j\kappa A_2(z), \\ \frac{\partial}{\partial z} A_2(z) &= -j\kappa A_1(z). \end{aligned}$$

Therefore:

$$\mathbf{M}_A = \begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}.$$

b) Eq. (3) describes the coupling of the amplitudes A_1 and A_2 of the individual waveguides. Alternatively, the system can be analyzed by considering the eigenmodes of the coupled waveguides. These eigenmodes do not interact with each other and can hence be found by transforming Eq. (3) to a basis in which $\mathbf{M}_A$ can be represented as a diagonal matrix. The entries in this diagonal matrix correspond to the eigenvalues of $\mathbf{M}_A$, and the corresponding eigenvectors represent the eigenmodes of the coupled waveguides.
Calculate the eigenvalues $\lambda_{e,o}$ and the normalized eigenvectors $\mathbf{v}_{e,o}$ of the matrix $\mathbf{M}_A$. In this notation, the subscript "e" refers to the eigenstate with even symmetry, i.e., the mode fields in waveguide 1 and 2 are superimposed with equal phase, and the subscript "o" refers to the eigenstate with odd symmetry, where the mode fields are superimposed with a phase shift of π.
Sketch the eigenstates of the coupled waveguides that correspond to λ_e and λ_o.

Solution: An eigenvector and the associated eigenvalue has to fulfill

$$\mathbf{M}_A \mathbf{v}_{e,o} = \lambda_{e,o} \mathbf{v}_{e,o}.$$

This leads to the eigenvalues $\lambda_{e,o} = \pm\kappa$, and the normalized eigenvectors $\mathbf{v}_e = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_o = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. A sketch of the modes is displayed on the Fig. (2).

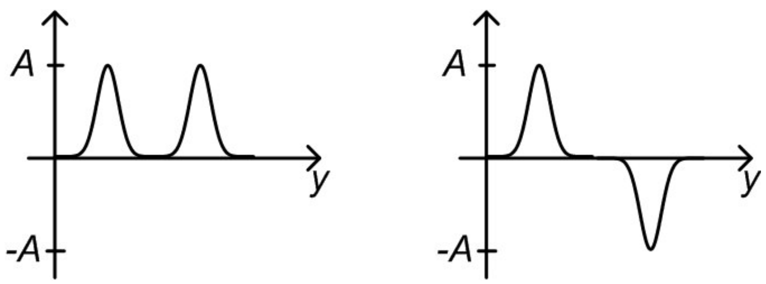

Figure 2: Mode profiles corresponding to the two eigenvectors

c) Using the result of b), Eq. (3) can be rewritten in the form

$$\frac{\partial}{\partial z} \mathbf{B}(z) = -j \mathbf{M}_B \mathbf{B}(z)$$

where $\mathbf{B}(z) = (B_e(z), B_o(z))^T$ and where $\mathbf{M}_B = \operatorname{diag}\{\lambda_e, \lambda_o\}$. Solve the decoupled differential equations and calculate $B_e(z)$ and $B_o(z)$ for given initial values $B_e(0)$ and $B_o(0)$. Assume that light is only launched in the left waveguide, i.e., $A_1(0) = A_0$ and $A_2(0) = 0$. Calculate the corresponding amplitudes $B_e(0)$ and $B_o(0)$ of the eigenstates of the coupled waveguides. Sketch the intensity profile in the two waveguides after propagation distances of $\kappa z = \frac{\pi}{4}$, $\kappa z = \frac{\pi}{2}$ and $\kappa z = \pi$.

Solution: First transform $\mathbf{A}(z) = \begin{pmatrix} A_1(z) \\ A_2(z) \end{pmatrix}$ into the coordinate system of the eigenvectors:

$$\begin{pmatrix} A_1(z) \\ A_2(z) \end{pmatrix} = B_e(z)\mathbf{v}_e + B_o(z)\mathbf{v}_o = B_e(z)\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B_o(z)\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} A_1(z) \\ A_2(z) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} B_e(z) + B_o(z) \\ B_e(z) - B_o(z) \end{pmatrix}.$$

From $A_1(0) = A_0$ and $A_2(0) = 0$ it follows: $B_e(0) = B_o(0) = \frac{1}{\sqrt{2}} A_0$. The provided matrix equation gives us decoupled differential equations for B_e and B_o.

$$\frac{\partial}{\partial z} \begin{pmatrix} B_e(z) \\ B_o(z) \end{pmatrix} = -j \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} B_e(z) \\ B_o(z) \end{pmatrix},$$

$$\frac{\partial}{\partial z} B_e(z) = -j\kappa B_e(z) \Rightarrow B_e(z) = B_e(0)\exp(-j\kappa z) = \frac{1}{\sqrt{2}} A_0 \exp(-jkz),$$

$$\frac{\partial}{\partial z} B_o(z) = -j\kappa B_o(z) \Rightarrow B_o(z) = B_o(0)\exp(j\kappa z) = \frac{1}{\sqrt{2}} A_0 \exp(jkz).$$

From here, we can get $A_1(z)$ and $A_2(z)$:

$$A_1(z) = \frac{1}{\sqrt{2}}(B_e(z) + B_o(z)) = \frac{A_0}{2}\left(\exp(-j\kappa z) + \exp(j\kappa z)\right) = A_0 \cos(\kappa z),$$
$$A_2(z) = \frac{1}{\sqrt{2}}(B_e(z) - B_o(z)) = \frac{A_0}{2}\left(\exp(-j\kappa z) - \exp(j\kappa z)\right) = -jA_0 \sin(\kappa z).$$

Sketches of the intensity profiles for $\kappa z = \frac{\pi}{4}$, $\kappa z = \frac{\pi}{2}$ and $\kappa z = \pi$ are displayed on the Fig. (3).

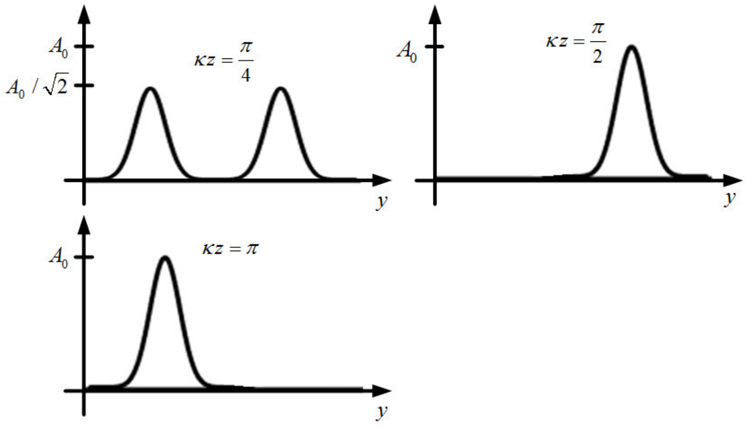

Figure 3: Intensity profiles for $\kappa z = \frac{\pi}{4}$, $\kappa z = \frac{\pi}{2}$ and $\kappa z = \pi$.

Questions and Comments:

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